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Contact problem for a two-layered cylinder*

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Контактная задача для двухслойного цилиндра***

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Introduction. The investigation of the contact problems for cylindrical bodies is urgent due to the engineering contact strength analysis on shafts, cores and pipe-lines. In the present paper, a new contact problem of elastostatics on the interaction between a rigid band and an infinite two-layered cylinder, which consists of an internal continuous cylinder and an outer hollow one, with a frictionless contact between the cylinders, is studied. The outer cylindrical band of finite length is press fitted. By using a Fourier integral transformation, the problem is reduced to an integral equation with respect to the unknown contact pressure.
Materials and Methods. Different combinations of linearly elastic materials of the composite cylinder are considered. Asymptotics of the symbol function of the integral equation kernel at zero and infinity is analyzed. This plays an important role for the application of the analytical solution methods. A key dimensionless geometric parameter is introduced, and a singular asymptotic technique is employed to solve the integral equation.
Research Results. On the basis of the symbol function properties, a special easily factorable approximation being applicable in a wide variation range of the problem parameters is suggested. The Monte-Carlo method is used to determine the approximation parameters. The asymptotic formulas are derived both for the contact pressure, and for its integral characteristic. Calculations are made for different materials and for various relative thickness of the cylindrical layer including thin-walled layers.
Discussion and Conclusions. The asymptotic solutions are effective for relatively wide bands when the contact zone length is bigger than the diameter of the composite cylinder. It is significant that the method is applicable also for those cases

Введение. Актуальность исследования контактных задач для цилиндрических тел обусловлена необходимостью проведения инженерных расчетов на контактную прочность валов, стержней и трубопроводов. В настоящей работе изучается новая контактная задача статической теории упругости о взаимодействии жесткого бандажа с бесконечным двухслойным цилиндром, состоящим из внутреннего сплошного и внешнего полого цилиндров, между которыми выполняются условия гладкого контакта. Наружный цилиндрический бандаж посажен с натягом и имеет конечную длину. При помощи интегрального преобразования Фурье задача сводится к интегральному уравнению относительно неизвестного контактного давления.
Материалы и методы. Рассматриваются разные комбинации линейно-упругих материалов составного цилиндра. Исследуется асимптотика функции-символа ядра интегрального уравнения в нуле и бесконечности, играющая важную роль для использования аналитических методов решения. Для решения интегрального уравнения вводится основной безразмерный геометрический параметр и применяется сингулярный асимптотический метод.
Результаты исследования. В соответствии со свойствами функции-символа предложена специальная легко факторизуемая аппроксимация этой функции, пригодная в широком диапазоне изменения параметров задачи. При помощи метода Монте-Карло рассчитаны параметры этой аппроксимации. Получены асимптотические формулы как для контактных давлений, так и для их интегральной характеристики. Расчеты сделаны для разных материалов и относительных толщин цилиндрического слоя, в том числе для тонкостенных слоев.
Обсуждение и заключения. Полученные асимптотические решения эффективны для относительно широких бандажей, когда размер области контакта превышает диаметр составного цилиндра. Важно, что используемый метод остается применимым и для случаев, когда внешний ци-

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when the outer cylindrical layer is treated as a cylindrical shell. The asymptotic solutions can be recommended to engineers for the contact strength analysis of the elastic barrels with a flexible coating of another material.

линдрический слой можно рассматривать как цилиндрическую оболочку. Асимптотические решения можно рекомендовать инженерам для анализа контактной прочности упругих деталей цилиндрической формы с упругим покрытием из другого материала.

Keywords: elasticity theory, contact problems, composite cylinder, approximation, asymptotics.

Ключевые слова: теория упругости, контактные задачи, составной цилиндр, аппроксимация, асимптотика.

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Introduction. A dynamic contact problem for a prestressed elastic liquid-filled cylinder is studied in the paper [1]. Static contact problems for homogeneous elastic cylindrical bodies were considered in the papers [2-6] using regular and singular asymptotic methods. It was found [4] that for cylindrical bodies, kernel symbols of integral equations of the contact problems are characterized by a more complex asymptotic behavior at zero and infinity than, for example, in contact problems for resilient strip. This required applying complicated approximations of the symbols easily factored by functions when using the singular asymptotic method. The approximation for hollow thin-walled cylinders is particularly complicated [6]. The suggested approximation [6] is advantageous even for the cases when a thin-walled elastic cylinder can be considered as a cylindrical shell [7]. The contact problem on the interaction of an elastic ring and a flexible cylinder was studied [8]. The elastic cylinder wear was analyzed in the paper [9]. This work objective is to obtain a solution to the contact problem for a composite two-layer elastic cylinder on the basis of a singular asymptotic technique and an effective approximation of the kernel symbol of the integral equation.

Materials and Methods. In the cylindrical coordinates *r, z* (with axisymmetry), we consider an infinite elastic composite cylinder of outer radius *R* which consists of an inner solid cylinder of radius *R1 <R* with elastic parameters ν_1, G_1 (Poisson's ratio and shear modulus) and an outer cylindrical layer with elastic parameters ν, G. Between the layer and the inner cylinder, the sliding binding conditions are met. We consider the contact problem on the interaction of the described composite cylinder and a rigid band in the domain $|z|\le a$. For the given band preload δ, it is required to estimate contact pressures $\sigma_r = -q(z) \quad (|z| \le a)$. Using the integral Fourier transform to solve the properly mixed (contact) boundary-value problem for Lame's elastic equilibrium equations and introducing non-dimensional notations (the primes are further omitted)

$$\lambda = \frac{R}{a}, \quad \delta' = \frac{\delta}{a}, \quad \zeta' = \frac{z}{a}, \quad q'(\zeta') = \frac{q(\zeta)(1-\nu)}{G}, \quad \varepsilon = \frac{G_1}{G}, \quad k = \frac{R_1}{R} < 1 , \tag{1}$$

we obtain the following integral equation for $q(\zeta)$:

$$\int_{-1}^{1} q(\xi) k\left(\frac{\zeta-\xi}{\lambda}\right) d\xi = \pi\delta \quad (|\zeta| \le 1), \qquad k(t) = \int_0^\infty L(u)\cos(ut)\,du, \tag{2}$$

where the kernel symbol takes the form

$$L(u) = -\frac{d_1}{d} I_1 - \frac{d_2}{d} K_1, \tag{3}$$

$$\begin{aligned} d = {} & (A_{55}A_{66} - A_{56}A_{65})[(A_{12}A_{31} - A_{11}A_{32})(A_{23}A_{44} - A_{24}A_{43}) - \\ & - A_{13}A_{31}(A_{22}A_{44} - A_{24}A_{42}) + A_{14}A_{31}(A_{22}A_{43} - A_{23}A_{42}) + \\ & + A_{13}A_{32}(A_{21}A_{44} - A_{24}A_{41}) - A_{14}A_{32}(A_{21}A_{43} - A_{23}A_{41})] + \\ & + A_{35}A_{56}[(A_{23}A_{44} - A_{24}A_{43})(A_{12}A_{61} - A_{11}A_{62}) + \\ & + (A_{22}A_{44} - A_{24}A_{42})(A_{11}A_{63} - A_{13}A_{61}) + (A_{22}A_{43} - A_{23}A_{42})(A_{14}A_{61} - A_{11}A_{64}) + \\ & + (A_{21}A_{44} - A_{24}A_{41})(A_{13}A_{62} - A_{12}A_{63}) + (A_{21}A_{43} - A_{23}A_{41})(A_{12}A_{64} - A_{14}A_{62}) + \\ & + (A_{21}A_{42} - A_{22}A_{41})(A_{14}A_{63} - A_{13}A_{64})], \end{aligned}$$

$$\begin{aligned} d_1 = {} & (A_{13}A_{44} - A_{14}A_{43})[A_{32}(A_{56}A_{65} - A_{55}A_{66}) - A_{35}A_{56}A_{62}] - \\ & - A_{35}A_{56}[A_{12}(A_{43}A_{64} - A_{44}A_{63}) - A_{42}(A_{13}A_{64} - A_{14}A_{63})], \end{aligned}$$

$$d_2 = -(A_{13}A_{44} - A_{14}A_{43})[A_{31}(A_{56}A_{65} - A_{55}A_{66}) - A_{35}A_{56}A_{61}] +$$
$$+A_{35}A_{56}[A_{11}(A_{43}A_{64} - A_{44}A_{63}) - A_{41}(A_{13}A_{64} - A_{14}A_{63})],$$

$$A_{11} = uI_0 - 2(1-\nu)I_1, \quad A_{12} = -uK_0 - 2(1-\nu)K_1, \quad A_{13} = uI_1, \quad A_{14} = -uK_1,$$

$$A_{21} = (3-2\nu)uI_0 - (u^2 + 4(1-\nu))I_1, \quad A_{22} = -(3-2\nu)uK_0 - (u^2 + 4(1-\nu))K_1,$$

$$A_{23} = u(I_1 - uI_0), \quad A_{24} = -u(K_1 + uK_0),$$

$$A_{31} = 2(1-\nu)I_1^*, \quad A_{32} = 2(1-\nu)K_1^*, \quad A_{35} = -4(1-\nu_1)I_1^*,$$

$$A_{41} = ukI_0^* - 2(1-\nu)I_1^*, \quad A_{42} = -ukK_0^* - 2(1-\nu)K_1^*, \quad A_{43} = uI_1^*, \quad A_{44} = -uK_1^*,$$

$$A_{55} = ukI_0^* - 2(1-\nu_1)I_1^*, \quad A_{56} = uI_1^*,$$

$$A_{61} = (3-2\nu)uI_0^* - (u^2k + 4(1-\nu)k^{-1})I_1^*,$$

$$A_{62} = -(3-2\nu)uK_0^* - (u^2k + 4(1-\nu)k^{-1})K_1^*,$$

$$A_{63} = uk^{-1}I_1^* - u^2I_0^*, \quad A_{64} = -uk^{-1}K_1^* - u^2K_0^*,$$

$$A_{65} = -\varepsilon[(3-2\nu_1)uI_0^* - (u^2k + 4(1-\nu_1)k^{-1})I_1^*], \quad A_{66} = -\varepsilon u(I_1^*k^{-1} - uI_0^*),$$

$$I_n = I_n(u), \quad K_n = K_n(u), \quad I_n^* = I_n(uk), \quad K_n^* = K_n(uk), \quad n = 0,\ 1.$$

Here, $I_n(u)$, $K_n(u)$ are modified Bessel functions. The dimensionless parameter λ characterizes a relative width of the contact region.

Function $L(u)$ at zero and infinity behaves as follows:

$$\lim_{u\to 0} L(u) = L(0) = \frac{(\nu-1)(1+\varepsilon-\nu_1+\varepsilon\nu_1) + k^2\varepsilon_1}{2(\nu-1)[(\nu+1)(1+\varepsilon-\nu_1+\varepsilon\nu_1) + k^2\varepsilon_1]}, \ \varepsilon_1 = (\nu+1)(\nu_1-1) - \varepsilon(\nu-1)(\nu_1+1). \quad (4)$$

$$L(u) = \frac{1}{u} + \frac{D}{u^2} + o(u^{-2}) \quad (u \to +\infty), \quad D = 1-2\nu.$$

At k=0, value $L(0)$ is the same as the known value for the homogeneous solid cylinder [4].

To solve the equation (2), we apply the singular asymptotic method [3,4] effective for sufficiently small values of λ. To apply the Wiener-Hopf technique [10], we used an easily factorable approximation of the function $L(u)$ (3) by the expression

$$L^*(u) = \frac{\sqrt{u^2+B^2}}{u^2+C^2}\exp\left(\frac{D}{\sqrt{u^2+10^4}}\right)\frac{u^2+A^2G^2}{u^2+G^2}, \quad C^2 = \frac{A^2B}{L(0)}\exp\left(\frac{D}{10^2}\right). \quad (5)$$

At calculations, two cases were taken: 1) iron inside, zinc outside (ε=2.126, ν=0.27, ν_1=0.28); 2) aluminum inside, zinc outside (ε=0.779, ν=0.27, ν_1=0.34). Table 1 gives the approximation parameters values (5), its relative error on the real axis θ (%) calculated using the Monte Carlo method for different relative thicknesses of the outer layer k.

Table 1

Approximation parameters

k	A	B	G	θ	A	B	G	θ
	Iron inside zinc				Aluminum inside zinc			
0.01	1.230	1.549	4.638	2.5	0.451	6.113	3.394	3
0.03	1.291	1.563	4.042	2.5	0.888	5.977	3.029	3
0.05	1.296	1.851	5.006	3	2.399	1.467	2.524	5
0.07	1.258	2.057	6.376	3	0.720	7.945	2.512	7
0.09	1.024	4.360	92.381	5	1.481	9.890	9.369	5
0.099	3.540	24.178	5.347	7	1.178	5.535	235.946	10

Considering, as follows from (4),

$$\lim_{k\to 1} L(0)=\frac{1-\nu_1}{2\varepsilon(1-\nu)(1+\nu_1)}, \tag{6}$$

then, in the case of small ε, the approximation (5) for thin outer layers is to be complicated by increase in number of the parameters included into it.

Research Results. As a result of applying the Wiener-Hopf technique, the principal term of the asymptotic solution to the integral equation (2) for small λ can be constructed in the form of

$$q(\zeta)=\frac{\delta}{\lambda}\left[\omega\left(\frac{1+\zeta}{\lambda}\right)+\omega\left(\frac{1-\zeta}{\lambda}\right)-\frac{1}{L(0)}\right] \quad (|x|\le 1), \tag{7}$$

$$\omega(s)=\frac{W(s)+I(s)}{\sqrt{L(0)}},\quad I(s)=-\frac{D}{\pi}\int_0^s W(s-\tau)K_0(10^2\tau)d\tau,$$

$$W(s)=\frac{\exp(-Bs)}{\sqrt{\pi s}}+\frac{C}{\sqrt{B}}\operatorname{erf}(\sqrt{Bs})+\left(\frac{1}{A}-1\right)Q(AG,s),$$

$$Q(F,s)=\frac{F-C}{\sqrt{B-F}}\exp(-Fs)\operatorname{erf}(\sqrt{(B-F)s})+\frac{C}{\sqrt{B}}\operatorname{erf}(\sqrt{Bs}).$$

Here, erf(x) is error function integral.

For the integral characteristic of the solution

$$P=\int_{-1}^{1} q(\zeta)d\zeta \tag{8}$$

on the basis of the formulas (7), we obtain the expression

$$\frac{P}{\delta}=\frac{2}{\sqrt{L(0)}}\left[Z\left(\frac{2}{\lambda}\right)+J\left(\frac{2}{\lambda}\right)\right]-\frac{2}{\lambda L(0)},\quad J(s)=-\frac{D}{\pi}\int_0^s Z(s-\tau)K_0(10^2\tau)d\tau, \tag{9}$$

$$Z(s)=\frac{C}{\sqrt{B}}\left[\left(s-\frac{1}{2B}+\frac{1}{C}\right)\operatorname{erf}(\sqrt{Bs})+\sqrt{\frac{s}{\pi B}}\exp(-Bs)\right]+\left(\frac{1}{A}-1\right)T(AG,s),$$

$$T(F,s)=\frac{C}{\sqrt{B}}\left[\left(s-\frac{1}{2B}+\frac{1}{C}-\frac{1}{F}\right)\operatorname{erf}(\sqrt{Bs})+\sqrt{\frac{s}{\pi B}}\exp(-Bs)\right]-$$

$$-\left(1-\frac{C}{F}\right)\frac{\exp(-Fs)}{\sqrt{B-F}}\operatorname{erf}(\sqrt{(B-F)s}).$$

Calculations show that the asymptotics (7), (9) error for λ<1 does not exceed (5+θ) %, where θ is the approximation (5) error.

Table 2 shows the values of the integral characteristic $P\delta^{-1}$ calculated from the formulas (9) at different values of k and λ.

Table 2

Values of $P\delta^{-1}$

λ=	2	1	0.5	0.25	2	1	0.5	0.25
k	Iron inside zinc				Aluminum inside zinc			
0.1 [,]	3.25	5.77	10.9	21.1	3.22	5.71	10.8	20.9
0.03	3.37	6.02	11.4	22.2	3.20	5.65	10.7	20.7
0.05	3.67	6.65	12.7	24.8	3.09	5.51	10.5	20.4
0.07	4.22	7.78	15.0	29.4	2.96	5.33	10.2	19.9
0.09	5.60	10.3	19.6	38.2	2.93	5.28	9.99	19.4
0.099	6.91	12.3	23.2	44.9	2.99	5.31	9.94	19.2

Discussion and Conclusions. As Table 2 shows, with decrease in λ, the integral characteristic of the contact pressures increases, which is associated with the extension of the contact area. For the case of a stronger material inside zinc (iron), contact pressures are higher than for aluminum inside zinc.

With thinning the zinc layer around the iron (with increasing *k*), the contact pressures increase essentially. This is hardly in evidence during thinning the layer of zinc around aluminum, since the modulus of longitudinal elasticity (and also shear modulus) of aluminum is slightly less than that of zinc. The asymptotics found can be recommended to engineers for analyzing the contact strength characteristics of the coated cylindrical parts.

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