

INFORMATION TECHNOLOGY, COMPUTER SCIENCE AND MANAGEMENT ИНФОРМАТИКА, ВЫЧИСЛИТЕЛЬНАЯ ТЕХНИКА И УПРАВЛЕНИЕ



UDC 519.7

Original Empirical Research

<https://doi.org/10.23947/2687-1653-2025-25-2-152-164>

Approximate Synthesis of H_∞ – Controllers in Nonlinear Dynamic Systems over a Semi-Infinite Time Period

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Abstract

Introduction. Problems and methods of finding H_∞ – control are the basis of modern control theory. They are actively used to develop robust controllers, especially in aircraft control systems under limited external actions. These methods allow for adapting control systems to changing environmental conditions, which is critically important for providing the reliability and safety of aircraft operation. Current research is aimed at improving approaches to the synthesis of controllers covering both linear and nonlinear dynamic systems. In this context, special attention is paid to the integration of new mathematical methods, such as linear matrix inequalities and frequency analysis, which allows for optimizing the system response to various external actions and providing protection against unexpected conditions. It is important to note that, despite the progress made in this area, significant problems remain unsolved regarding the analysis and synthesis of controllers for nonlinear systems. This necessitates further research and development in this promising area. In this paper, in order to fill the existing gap, sufficient conditions for the existence of control for one of the frequently encountered classes of nonlinear systems are formulated and proven, which will then be used as a theoretical basis for developing approximate algorithms for finding it.

Materials and Methods. The basic research tool was the H_∞ – control synthesis methods based on the minimax approach, which consisted in finding the control law under the worst external action. In this context, it was proposed to prove sufficient conditions for the existence of control using the extension principle. However, due to the computational difficulties that might arise when applying those conditions, it was decided to simplify the initial formulation of the problem. The simplification process was performed by approximate replacing the nonlinear system with another nonlinear system, which was similar in structure to the linear one, using the factorization procedure. This approach made it possible to use the solution of the Riccati equation, whose coefficients depended on the state vector, for the synthesis of controllers. To solve model examples and applied problems, a software package was developed using the MATLAB mathematical package.

Results. The article solved the problem of synthesis of H_∞ – control of the state of nonlinear continuous dynamic systems, linear in control and disturbance. Sufficient conditions for the existence of H_∞ – control were formulated and proved on the basis of the extension principle. An approximate method was proposed that provided solving the problem of finding control laws for dynamic systems that were nonlinear in state, similar to the methods used for linear systems. Analytical solutions were found for two model examples, which were illustrated by graphs of transient processes to demonstrate the results of numerical modeling of the considered nonlinear dynamic systems in the presence of external actions.

Discussion and Conclusion. The proposed approximate algorithm for synthesizing state and output controllers guarantees the required quality of transient processes and asymptotic stability of closed nonlinear control systems. This significantly expands the class of dynamic systems for which it is possible to synthesize controllers capable of resisting various external actions. The methods presented in this paper can be effectively applied to solve a variety of control problems, including the design of autopilots and automatic navigation systems for aircraft, even under conditions of limited external actions.

Keywords: H_∞ – control, nonlinear dynamic system, semi-infinite period of time, feedback control, controller synthesis

Acknowledgements. The authors would like to thank Irina Anatolyevna Kudryavtseva, Cand.Sci. (Phys.-Math.), Associate Professor, for assistance in preparing the text of the article.

For citation. Pantelev AV, Yakovleva AA. Approximate Synthesis of H_∞ – Controllers in Nonlinear Dynamic Systems over a Semi-Infinite Time Period. *Advanced Engineering Research (Rostov-on-Don)*. 2025;25(2):152–164. <https://doi.org/10.23947/2687-1653-2025-25-2-152-164>

Оригинальное эмпирическое исследование

Приближенный синтез H_∞ – регуляторов в нелинейных динамических системах на полубесконечном промежутке времени

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Аннотация

Введение. Задачи и методы нахождения H_∞ – управления являются основой современной теории управления и активно используются для разработки робастных регуляторов, особенно в системах управления летательными аппаратами под ограниченными внешними воздействиями. Эти методы позволяют адаптировать системы управления к изменяющимся условиям окружающей среды, что критически важно для обеспечения надежности и безопасности работы летательных аппаратов. Текущие исследования направлены на усовершенствование подходов к синтезу регуляторов, охватывающих как линейные, так и нелинейные динамические системы. В этом контексте особое внимание уделяется интеграции новых математических методов, таких как линейные матричные неравенства и частотный анализ, что позволяет оптимизировать отклик системы на различные внешние воздействия и гарантировать защиту от непредвиденных условий. Важно отметить, что, несмотря на достигнутые успехи в данной области, остаются нерешенными значительные проблемы, касающиеся анализа и синтеза регуляторов для нелинейных систем. Это создает необходимость в дальнейших исследованиях и разработках в этой перспективной области. В данной работе, с целью заполнения существующего пробела, сформулированы и доказаны достаточные условия существования управления для одного из часто встречающихся классов нелинейных систем, которые затем будут использоваться в качестве теоретического обоснования для разработки приближенных алгоритмов его нахождения.

Материалы и методы. В качестве основного инструмента исследования используются методы синтеза H_∞ – управления, основанные на минимаксном подходе, заключающемся в нахождении закона управления в условиях наихудшего внешнего воздействия. В этом контексте предлагается доказать достаточные условия существования управления, используя принцип расширения. Однако из-за вычислительных трудностей, которые могут возникнуть при применении этих условий, было решено упростить исходную постановку задачи. Процесс упрощения осуществлялся путем приближенной замены нелинейной системы на другую нелинейную систему, которая по своей структуре схожа с линейной, с помощью процедуры факторизации. Такой подход позволяет применять решение уравнения Риккати, коэффициенты которого зависят от вектора состояния, для синтеза регуляторов. Для решения модельных примеров и прикладных задач был разработан программный комплекс с использованием математического пакета MATLAB.

Результаты исследования. В статье решена проблема синтеза H_∞ – управления состоянием нелинейных непрерывных динамических систем, линейных по управлению и возмущению; сформулированы и на основе принципа расширения доказаны достаточные условия существования H_∞ – управления. Предложен приближенный метод, позволяющий решать задачу нахождения законов управления для динамических систем, нелинейных по состоянию, аналогичный методам, применяемым для линейных систем. Найден аналитические решения двух модельных примеров, которые проиллюстрированы графиками переходных процессов для демонстрации результатов численного моделирования рассмотренных нелинейных динамических систем в присутствии внешних воздействий.

Обсуждение и заключение. Предложенный приближенный алгоритм синтеза регуляторов по состоянию и выходу гарантирует необходимое качество переходных процессов и асимптотическую устойчивость замкнутых нелинейных систем управления. Это значительно расширяет класс динамических систем, для которых возможно синтезирование регуляторов, способных противостоять различным внешним воздействиям. Методы, изложенные в данной работе, могут быть эффективно применены для решения множества задач управления, включая проектирование автопилотов и автоматических навигационных систем для летательных аппаратов, даже в условиях ограниченного воздействия извне.

Ключевые слова: H_∞ – управление, нелинейная динамическая система, полубесконечный промежуток времени, управление с обратной связью, синтез регуляторов

Благодарности. Авторы выражают признательность доценту, кандидату физико-математических наук Кудрявцевой Ирине Анатольевне за помощь в подготовке текста статьи.

Для цитирования. Пантелеев А.В., Яковлева А.А. Приближенный синтез H_∞ – регуляторов в нелинейных динамических системах на полубесконечном промежутке времени. *Advanced Engineering Research (Rostov-on-Don)*. 2025;25(2):152–164. <https://doi.org/10.23947/2687-1653-2025-25-2-152-164>

Introduction. Methods of modern control theory play an important role in the development of complex aerospace systems, providing their efficient operation. To achieve high productivity, stability and efficiency of such systems, it is necessary to develop algorithms for synthesizing controllers capable of operating under conditions of uncertainty in the description of external influences. The modern foundation for their development include the state space method, frequency analysis, and the approach based on linear matrix inequalities [1]. To solve problems of finding optimal control, sufficient optimality conditions in the form of the Bellman equation and the relations following from it in special cases are usually applied. Linear matrix inequalities can be used to search for H_∞ – controllers. They determine the existence of a regulator that satisfies certain performance criteria and provides the stability of the system to external influences. These criteria are usually associated with a norm, which is a measure of the sensitivity of the system to external disturbances. These criteria are usually related to H_∞ – norm, which is a measure of the sensitivity of the system to external disturbances. The problem is to find a regulator that minimizes this norm, while providing the stability of the system and satisfying the quality criterion of control. The solution method is based on finding the extremum of a convex objective function, where the conditions are presented in the form of linear matrix inequalities [2]. By using this method, it is possible to reduce the solution of complex systems of linear and nonlinear algebraic matrix equations of a certain type to the solution of convex optimization problems. However, the solution of linear matrix inequalities can be difficult when considering complex technical problems.

An alternative method based on stochastic minimax is presented in the anisotropic theory of stochastic robust control described in [3]. The main idea in applying this method is that robustness in stochastic control is reached by explicitly including different noise distribution scenarios in a single performance indicator to be optimized. Statistical uncertainty is expressed through entropy, and the robust quality indicator is selected in such a way as to make it possible to quantify the system's ability to suppress the worst external impact. The application of such an approach to solving complex systems of interrelated equations requires the development and use of specialized algorithms.

It should be noted that methods of H_∞ – optimization are used to solve numerous different applied problems, such as aircraft [4], helicopter [5], quadcopter [6] and multi-agent systems [7] control, robot stabilization [8], rocket engine design [9], where, when compared to other controllers, these methods show good results and lower error values under limited disturbances. It is also worth mentioning their use in filtering problems [10], state vector estimation [11] and neural network design [12]. Thus, the development and advancement of H_∞ – optimization methods are highly topical issues for research. Previously, the authors considered the problems of synthesizing H_∞ – controller [13] and H_∞ – observer [14] for linear dynamic systems, for the solution of which sufficient optimality conditions based on the expansion principle were used. Their application made it possible to justify the synthesis procedures and, as a result, to form step-by-step algorithms for solving problems.

Despite significant achievements in this area, a number of problems related to the analysis and synthesis of controllers for nonlinear systems remain unsolved. In this regard, the paper considers the problem of synthesis of controllers for nonlinear dynamic systems, linear in control and disturbance, on a semi-infinite time interval. The research objective is to formulate and prove sufficient conditions for the existence of control. This will not only create a basis for new research and development, but also fill existing gaps in the field of knowledge. Specifically, the work provides for the use of sufficient conditions as a theoretical justification for the formulation of approximate control search algorithms for the class of dynamic systems under consideration. To test the efficiency of the proposed algorithm, two model examples will be solved.

Materials and Methods. Let there be a mathematical model of the control object:

$$\dot{x}(t) = f(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = \mathbf{0}, \quad (1)$$

and the model of the measuring system:

$$y(t) = C(x(t))x(t), \quad (2)$$

where $x \in R^n$ — state vector, $u \in R^q$ — control vector, $w \in R^p$ — external influences vector, $y \in R^m$ — output vector, $t \in T = [0, \infty)$ — current time, $\mathbf{0}$ — zero matrix-column of dimensions $(n \times 1)$. Assume that the continuously differentiable vector function $f(x)$ of dimensions $(n \times 1)$, as well as the matrix functions $B_1(x)$ of dimensions $(n \times p)$, $B_2(x)$ of dimensions $(n \times q)$, $C(x)$ of dimensions $(m \times n)$ are given. The model of the object is described by an equation that is nonlinear in state, but linear with respect to control and external influences.

It is implied that:

- a) $w(\cdot) \in L_2[0, \infty)$, $u(\cdot) \in L_2[0, \infty)$;
- b) $m \leq n$, $\text{rg } C(x) = m$;
- c) the origin of coordinates $x \equiv 0$ is the equilibrium point, e.g., $f(0) = 0$;
- d) $B_1(x) \neq 0$, $B_2(x) \neq 0$;

where O — zero matrix of corresponding dimensions.

An indicator describing the current behavior of the control object model (1) with measuring system (2) is defined:

$$\|z(t)\|^2 = y(t)^T S(x(t))y(t) + u^T(t)Q(x(t))u(t), \quad (3)$$

where for all $x \in R^n$ $Q(x) > 0$ — symmetric positive definite square matrix of order q , and $S(x) > 0$ — symmetric non-negative definite square matrix of order m . Functional (3) is quadratic in control, but non-quadratic in state.

Note that we are considering models of an object and a measuring system, the matrices in which depend on the state vector.

It is required to ensure the correctness of the condition:

$$\begin{aligned} \frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|w(t)\|_p^2 dt} &= \frac{\int_0^\infty [y(t)^T S(x(t))y(t) + u^T(t)Q(x(t))u(t)] dt}{\int_0^\infty \|w(t)\|_p^2 dt} = \\ &= \frac{\int_0^\infty [x^T(t)C^T(x(t))S(x(t))C(x(t))x(t) + u^T(t)Q(x(t))u(t)] dt}{\int_0^\infty w^T(t)P(x(t))w(t) dt} \leq \gamma^2, \end{aligned} \quad (4)$$

where $\forall x \in R^n$ $P(x) > 0$ — symmetric square matrix of order p , $\gamma > 0$ — some number. As an additional condition, the necessity of fulfilling the property of asymptotic stability of the closed system “object-controller” is considered. Note that it is important to find the smallest value of parameter γ^* that provides the preservation of the required properties of the closed system. This is possible only if the conditions of minimizing the numerator and maximizing the denominator of the expression are simultaneously fulfilled (4).

We rewrite condition (4) in the form:

$$\begin{aligned} I(u, w) &= \int_0^\infty [\|z(t)\|^2 - \gamma^2 \|w(t)\|_p^2] dt = \\ &= \int_0^\infty [x^T(t)C^T(x(t))S(x(t))C(x(t))x(t) + u^T(t)Q(x(t))u(t) - \gamma^2 w^T(t)P(x(t))w(t)] dt \leq 0. \end{aligned} \quad (5)$$

This means that it is required to provide that inequality (5) is satisfied while minimizing control costs under conditions of maximum counteraction of external influences (disturbances).

Sufficient conditions for the existence of H_∞ – controllers

Assume that function $V(x) \in C^1(R^n)$ is known. Let us define the function:

$$\begin{aligned} R(x, u, w) &= \left(\frac{\partial V(x)}{\partial x} \right)^T [f(x) + B_1(x)w + B_2(x)u] + x^T C^T(x)S(x)C(x)x + \\ &\quad + u^T Q(x)u - \gamma^2 w^T P(x)w, \end{aligned} \quad (6)$$

$$\text{where } \frac{\partial V(x)}{\partial x} = \left(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_n} \right)^T.$$

Theorem. If there exists function $V(x) \in C^1(R^n)$, satisfying conditions $V(0) = 0$ and

$$R(x, u^*(x), w^*(x)) = \min_u \max_w R(x, u, w) = 0 \quad \forall x \in R^n, \quad (7)$$

where

$$u^*(x) = -\frac{1}{2} Q^{-1}(x) B_2^T(x) \frac{\partial V(x)}{\partial x}, \quad w^*(x) = \frac{1}{2\gamma^2} P^{-1}(x) B_1^T(x) \frac{\partial V(x)}{\partial x}, \quad (8)$$

and function $V(x)$ is determined by solving a partial differential equation:

$$\begin{aligned} &\left(\frac{\partial V(x)}{\partial x} \right)^T f(x) - \frac{1}{4} \left(\frac{\partial V(x)}{\partial x} \right)^T B_2(x) Q^{-1}(x) B_2^T(x) \frac{\partial V(x)}{\partial x} + \\ &+ \frac{1}{4\gamma^2} \left(\frac{\partial V(x)}{\partial x} \right)^T B_1(x) P^{-1}(x) B_1^T(x) \frac{\partial V(x)}{\partial x} + x^T C^T(x)S(x)C(x)x = 0, \end{aligned} \quad (9)$$

then condition (4) is satisfied.

Proof. Suppose that the conditions of the theorem are satisfied. We find $\min_u \max_w R(x, u, w)$, applying the required conditions for an unconditional extremum, since no restrictions are imposed on the variables u, w :

$$\begin{aligned}\frac{\partial R(x, u, w)}{\partial u} &= B_2^T(x) \frac{\partial V(x)}{\partial x} - 2Q(x)u = 0, \\ \frac{\partial R(x, u, w)}{\partial w} &= B_1^T(x) \frac{\partial V(x)}{\partial x} - 2\gamma^2 P(x)w = 0.\end{aligned}$$

Solving the matrix equations, we obtain:

$$u^*(x) = -\frac{1}{2} Q^{-1}(x) B_2^T(x) \frac{\partial V(x)}{\partial x}, \quad w^*(x) = \frac{1}{2\gamma^2} P^{-1}(x) B_1^T(x) \frac{\partial V(x)}{\partial x},$$

where $u^*(x), w^*(x)$ — control structures of the object model and external influence (disturbance).

Since $\frac{\partial^2 R(x, u, w)}{\partial u^T \partial u} = 2Q(x) > 0 \quad \forall x \in R^n$ is true, then the sufficient conditions for the minimum in control are satisfied.

Also true are the sufficient conditions that guarantee the achievement of the maximum external influence w , since $\frac{\partial^2 R(x, u, w)}{\partial w^T \partial w} = -2\gamma^2 P(x) < 0 \quad \forall x \in R^n$.

In this case

$$R(x, u, w) = R(x, u^*(x), w^*(x)) - \gamma^2 [w - w^*(x)]^T P(x) [w - w^*(x)] + [u - u^*(x)]^T Q(x) [u - u^*(x)].$$

From this we get

$$R(x, u^*(x), w(x)) \leq R(x, u^*(x), w^*(x)) \leq R(x, u(x), w^*(x)), \quad (10)$$

i.e., the conditions for the presence of a saddle point are satisfied.

Assume that function $V(x) \in C^1(R^n)$ satisfies conditions $V(0) = 0$ and $R(x, u^*(x), w^*(x)) = 0$.

Then the relation that is fulfilled along the trajectories of system (1) is valid, namely:

$$\begin{aligned}& \left(\frac{\partial V(x(t))}{\partial x} \right)^T [f(x(t)) + B_1(x(t))w(x(t)) + B_2(x(t))u(x(t))] + \|z(t)\|^2 - \gamma^2 \|w(t)\|_p^2 = \\&= \frac{dV(x(t))}{dt} + \|z(t)\|^2 - \gamma^2 \|w(t)\|_p^2 = \underbrace{R(x(t), u^*(x(t)), w^*(x(t)))}_0 - \\& - \gamma^2 [w(t) - w^*(x(t))]^T P(x(t)) [w(t) - w^*(x(t))] + [u(t) - u^*(x(t))]^T Q(x(t)) [u(t) - u^*(x(t))].\end{aligned}$$

For $u = u^*(x)$, let us rewrite the left side of the inequality (10), i.e., $R(x, u^*(x), w(x)) \leq \underbrace{R(x, u^*(x), w^*(x))}_0$, in the form:

$$\frac{dV(x(t))}{dt} + \|z(t)\|^2 - \gamma^2 \|w(t)\|_p^2 \leq 0.$$

By integrating the left and right sides of the resulting inequality over the time interval from 0 to t_1 , we obtain:

$$V(x(t_1)) - V(x(0)) + \int_0^{t_1} \|z(t)\|^2 dt - \gamma^2 \int_0^{t_1} \|w(t)\|_p^2 dt \leq 0.$$

Since it is required to provide the fulfillment of the condition of asymptotic stability of the closed system, then $x(t_1) \rightarrow 0$ when $t_1 \rightarrow +\infty$, therefore $V(x(t_1)) \rightarrow V(0) = 0$. Since $x(0) = 0$ then $V(x(0)) \rightarrow V(0) = 0$. From this we can conclude that when $t_1 \rightarrow +\infty$ the inequality is valid:

$$\int_0^\infty \|z(t)\|^2 dt \leq \gamma^2 \int_0^\infty \|w(t)\|_p^2 dt,$$

indicating that condition (4) is satisfied, which was to be proved.

As a remark, we emphasize that when conditions $P(x) = E$, $\int_0^\infty \|w(t)\|^2 dt \leq 1$, i.e., the energy of external influences is

limited, an inequality of the form $\int_0^\infty \|z(t)\|^2 dt \leq \gamma^2$ is valid.

Approximate synthesis of H_∞ – state-based controllers using the SDRE method

Due to the nonlinearity of equation (9) and difficulties in obtaining its solution, a method based on the algebraic Riccati equation with coefficients dependent on the state vector is used for further analysis [15].

As a result of applying the factorization operation, we obtain a nonlinear system transformed to a structure similar to a linear one, with matrices dependent on the state vector.

It is known [16] that if $f(0) = 0$ and $f(x) \in C^1(R^n)$, then there exists matrix function $A(x)$, such that:

$$f(x) = A(x)x. \quad (11)$$

Notes

1. The factorization procedure for $n = 1$ is unique $\forall x \neq 0$, i.e., $A(x) = f(x) / x = a(x)$.
2. For $n > 1$, the factorization procedure yields a non-unique result [16]. For example, for $n = 2$, there are at least two options: $f(x) = A_1(x)x$ and $f(x) = A_2(x)x$, i.e., for a system of the form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} + B_1(x)w + B_2(x)u$$

we obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{f_1(x)}{x_1} & 0 \\ \frac{f_2(x)}{x_1} & 0 \end{pmatrix}}_{A_1(x)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + B_1(x)w + B_2(x)u \text{ or } \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{f_1(x)}{x_2} \\ 0 & \frac{f_2(x)}{x_2} \end{pmatrix}}_{A_2(x)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + B_1(x)w + B_2(x)u.$$

3. If there are two parametrization options, i.e., $f(x) = A_1(x)x = A_2(x)x$, then there is an infinite family of options of the form [16]: $A(x, \alpha) = \alpha A_1(x) + (1 - \alpha)A_2(x) \forall \alpha$. The selection of parameter α allows for flexibility in designing the control system. The solution to the Riccati equation and the corresponding control become functions of this parameter.

As a result of factorization, the mathematical model of system (1) takes the form:

$$\dot{x}(t) = A(x(t))x(t) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = 0. \quad (12)$$

On the trajectories of system (12), functional (5) is specified.

It is assumed that system (12) is controllable and observable, i.e., $\forall x \in R^n$ the conditions are fulfilled point-to-point [16]:

$$\begin{aligned} \text{rg}[B_2(x) \ A(x)B_2(x) \ \dots \ A^{n-1}(x)B_2(x)] &= n, \\ \text{rg}[C^T(x) \ A^T(x)C^T(x) \ \dots \ (A^T(x))^{n-1}C^T(x)] &= n. \end{aligned}$$

Equation (9) takes the form:

$$\begin{aligned} &\left(\frac{\partial V(x)}{\partial x}\right)^T A(x)x - \frac{1}{4}\left(\frac{\partial V(x)}{\partial x}\right)^T B_2(x)Q^{-1}(x)B_2^T(x)\frac{\partial V(x)}{\partial x} + \\ &+ \frac{1}{4\gamma^2}\left(\frac{\partial V(x)}{\partial x}\right)^T B_1(x)P^{-1}(x)B_1^T(x)\frac{\partial V(x)}{\partial x} + x^T C^T(x)S(x)C(x)x = 0. \end{aligned} \quad (13)$$

Assume that

$$\frac{\partial V(x)}{\partial x} = 2K_2(x)x, \quad (14)$$

where $K_2(x) > 0$ unknown matrix function (for fixed $x \in R^n$, matrix $K_2(x)$ is a symmetric positive definite numerical matrix). Thus, an assumption is made not about the type of function $V(x)$, but only about the structure of its partial derivative.

Then, the control structures of the object and disturbance take the form:

$$\begin{aligned} u^*(x) &= -\frac{1}{2}Q^{-1}(x)B_2^T(x)\frac{\partial V(x)}{\partial x} = -Q^{-1}(x)B_2^T(x)K_2(x)x, \\ w^*(x) &= \frac{1}{2\gamma^2}P^{-1}(x)B_1^T(x)\frac{\partial V(x)}{\partial x} = \frac{1}{\gamma^2}P^{-1}(x)B_1^T(x)K_2(x)x. \end{aligned} \quad (15)$$

Equation (13) takes the form:

$$\begin{aligned} &2x^T K_2(x)A(x)x - x^T K_2(x)B_2(x)Q^{-1}(x)B_2^T(x)K_2(x)x + \\ &+ \frac{1}{\gamma^2}x^T K_2(x)B_1(x)P^{-1}(x)B_1^T(x)K_2(x)x + x^T C^T(x)S(x)C(x)x = 0, \end{aligned}$$

or

$$x^T [2K_2(x)A(x) - K_2(x)B_2(x)Q^{-1}(x)B_2^T(x)K_2(x) + \frac{1}{\gamma^2}K_2(x)B_1(x)P^{-1}(x)B_1^T(x)K_2(x) + C^T(x)S(x)C(x)]x = 0.$$

Applying the transpose operation, we obtain:

$$x^T [2A^T(x)K_2(x) - K_2(x)B_2(x)Q^{-1}(x)B_2^T(x)K_2(x) + \frac{1}{\gamma^2}K_2(x)B_1(x)P^{-1}(x)B_1^T(x)K_2(x) + C^T(x)S(x)C(x)]x = 0.$$

Summing up the last two expressions, we arrive at the equality:

$$x^T [2K_2(x)A(x) + 2A^T(x)K_2(x) - 2K_2(x)B_2(x)Q^{-1}(x)B_2^T(x)K_2(x) + \frac{2}{\gamma^2}K_2(x)B_1(x)P^{-1}(x)B_1^T(x)K_2(x) + 2C^T(x)S(x)C(x)]x = 0,$$

or finally:

$$x^T [K_2(x)A(x) + A^T(x)K_2(x) - K_2(x)B_2(x)Q^{-1}(x)B_2^T(x)K_2(x) + \frac{1}{\gamma^2}K_2(x)B_1(x)P^{-1}(x)B_1^T(x)K_2(x) + C^T(x)S(x)C(x)]x = 0. \quad (16)$$

Note that due to the dependence of all matrices on the state vector, the equality of the zero matrix in square brackets does not follow from (16).

By analogy with the case of a linear stationary system, it is proposed to solve the algebraic Riccati equation, all matrices in which are functions of the system's state vector (State Dependent Riccati Equation, SDRE). In this case, a positive definite solution to the Riccati equation is sought, generating a control law that guarantees that the system will be asymptotically stable in the vicinity of the equilibrium position. To check this property, the stability criterion by the roots of the characteristic equation, checked point-to-point, or the Routh–Hurwitz criterion, is used.

In the problem under consideration, it is proposed to solve the equation:

$$K_2(x)A(x) + A^T(x)K_2(x) - K_2(x)[B_2(x)Q^{-1}(x)B_2^T(x) - \frac{1}{\gamma^2}B_1(x)P^{-1}(x)B_1^T(x)]K_2(x) + C^T(x)S(x)C(x) = O, \quad (17)$$

i.e., to look for matrix $K_2(x) > 0$, that satisfies the Riccati equation, whose coefficients depend on x . Equation (17) is solved repeatedly for fixed $x \in R^n$. The coordinates of the state vector are determined in the process of integrating the differential equation (12) together with the controls of the object and the disturbances:

$$u^*(x) = -Q^{-1}(x)B_2^T(x)K_2(x)x, \quad w^*(x) = \frac{1}{\gamma^2}P^{-1}(x)B_1^T(x)K_2(x)x. \quad (18)$$

In this case, the solution to the Riccati equation must be such that the criterion $\sigma\{A(x) + [\frac{1}{\gamma^2}B_1(x)P^{-1}(x)B_1^T(x) - B_2(x)Q^{-1}(x)B_2^T(x)]K_2(x)\} \subset C^- \quad \forall x \in R^n$, is satisfied, where σ^- matrix spectrum, C^- — open left half-plane of the complex plane. Note that the stability criterion of a closed system can be replaced by checking the point-to-point fulfillment of the Routh-Hurwitz criterion.

Algorithm for approximate synthesis of H_∞ – controllers of state

Step 1. Set parameter $\gamma > 0$.

Step 2. Find the solution to the equation:

$$\dot{x}(t) = A(x(t))x(t) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = 0,$$

with controls

$$u(t) = u^*(x(t)) = -Q^{-1}(x(t))B_2^T(x(t))K_2(x(t))x(t),$$

$$w(t) = w^*(x(t)) = \frac{1}{\gamma^2}P^{-1}(x(t))B_1^T(x(t))K_2(x(t))x(t),$$

one of the numerical methods of integration with constant step h (explicit Euler method, Euler-Cauchy method, Adams-Bashforth, Milne, Hamming methods of various orders).

In this case, for each of the discrete moments of time $t_i = ih$, $i = 0, 1, 2, \dots$, solve the Riccati equation:

$$K_2(x)A(x) + A^T(x)K_2(x) - K_2(x)[B_2(x)Q^{-1}(x)B_2^T(x) - \frac{1}{\gamma^2}B_1(x)P^{-1}(x)B_1^T(x)]K_2(x) + C^T(x)S(x)C(x) = O,$$

for $x = x(t_i)$. As a result, find matrix $K_2(x)$ and use it to form control laws.

Step 3. Find the minimum γ^* . To do this, it is required to consistently decrease γ until the stability property of solutions of the differential equation

$$\dot{x}(t) = [A(x(t)) + \frac{1}{\gamma^2}B_1(x(t))P^{-1}(x(t))B_1^T(x(t)) - B_2(x(t))Q^{-1}(x(t))B_2^T(x(t))]K_2(x(t))x(t),$$

describing the dynamics of the system with the obtained controls, remains valid.

Research Results. To test the efficiency of the proposed approximate algorithm for synthesizing H_∞ – state-based controllers, two model examples were solved.

Model example No. 1. A one-dimensional case is considered, when equations (1), (2) and functional (5) have the form:

$$\dot{x}(t) = f(x) + B_1(x)w + B_2(x)u = A(x)x + B_1(x)w + B_2(x)u,$$

$$y = C(x)x,$$

$$I = \int_0^\infty [S(x(t))x^2(t) + Q(x(t))u^2(t) - \gamma^2 P(x(t))w^2(t)] dt.$$

Solution. The control structures follow from (18):

$$u^*(x) = -\frac{1}{Q(x)} B_2(x) K_2(x) x, \quad u^*(x) = -\frac{1}{Q(x)} B_2(x) K_2(x) x,$$

and equation (17) has the form:

$$2A(x)K_2(x) - K_2^2(x) \left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right] + C^2(x)S(x) = 0.$$

Let us write the resulting quadratic equation in canonical form:

$$\left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right] K_2^2(x) - 2A(x)K_2(x) - C^2(x)S(x) = 0.$$

The solution is as follows:

$$K_2(x) = \frac{2A(x) \pm \sqrt{4A^2(x) + 4C^2(x)S(x) \left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right]}}{2 \left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right]}.$$

Since $K_2 > 0$, then

$$K_2(x) = \frac{A(x) + \sqrt{A^2(x) + C^2(x)S(x) \left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right]}}{\left[\frac{B_2^2(x)}{Q(x)} - \frac{1}{\gamma^2 P(x)} B_1^2(x) \right]}.$$

Let us take a closer look at a particular case:

$$\dot{x}(t) = x - x^3 + w + u = (1 - x^2)x + w + u,$$

$$y = 2x,$$

$$I = \int_0^\infty [x^2(t) + u^2(t) - \gamma^2 w^2(t)] dt,$$

where $A(x) = (1 - x^2)$, $B_1(x) = 1$, $B_2(x) = 1$, $C(x) = 2$, $Q(x) = 1$, $S(x) = 1$, $P(x) = 1$.

Then (omitting the dependence on x) we get:

$$2AK_2 - K_2^2 \left(1 - \frac{1}{\gamma^2} \right) + C^2 = 0 \rightarrow K_2^2 (1 - \gamma^{-2}) - 2AK_2 - C^2 = 0.$$

Roots of the quadratic equation:

$$K_2 = \frac{A \pm \sqrt{A^2 + C^2(1 - \gamma^{-2})}}{1 - \gamma^{-2}}, \quad \gamma \neq 1, \quad K_2 = -\frac{C^2}{2A}, \quad \gamma = 1.$$

Note that when considering the case $\gamma \neq 1$, it is not yet possible to exclude the extra roots, since $A(x)$ and $(1 - \gamma^{-2})$ can change the sign.

As a result, we obtain the control structures:

$$w^*(x) = \frac{1}{\gamma^2} B_1^T K_2 x = \frac{1}{\gamma^2} K_2 x, \quad u^*(x) = B_2^T K_2 x = -K_2 x.$$

In this case, the equation of the closed system has the form:

$$\dot{x} = [A - (1 - \gamma^{-2})K_2]x = \mp \sqrt{A^2 + C^2(1 - \gamma^{-2})} x.$$

To provide asymptotic stability, we take the minus sign, and in the expression for K_2 – plus. For this example, we get:

1) if $\gamma \neq 1$, then

$$K_2(x) = \frac{A(x) + \sqrt{A^2(x) + C^2(x)(1 - \gamma^{-2})}}{1 - \gamma^{-2}} = \frac{1 - x^2 + \sqrt{(1 - x^2)^2 + 4(1 - \gamma^{-2})}}{1 - \gamma^{-2}} > 0;$$

2) if functions $A(x)$, $C(x)$ are not equal to zero simultaneously, then discriminant $A^2(x) + C^2(x)(1 - \gamma^2)$.

Then,

$$A^2(x) + C^2(x) \geq C^2(x)\gamma^{-2}, \quad \gamma^{-2} \leq \frac{A^2(x) + C^2(x)}{C^2(x)}, \quad \gamma^2 \geq \frac{C^2(x)}{A^2(x) + C^2(x)}, \quad \gamma_{\min}^2(x) = \frac{C^2(x)}{A^2(x) + C^2(x)}.$$

For the example being solved $\gamma_{\min}^2(x) = \frac{C^2(x)}{A^2(x) + C^2(x)} = \frac{4}{(1-x^2)^2 + 4}$, i.e., for each current x , there is its own value $\gamma_{\min}^2(x)$.

For $\gamma = \gamma_{\min}$, $x = 0$, $x(0) = 0$ is satisfied (the condition of asymptotic stability is not satisfied, but $x(t) \equiv 0$ is valid).

For $\gamma = \gamma_{\min}$ we have

$$K_2(x) = \frac{A(x) + \sqrt{A^2(x) + C^2(x)(1 - \gamma^{-2})}}{1 - \gamma^{-2}} = \frac{1 - x^2 + \sqrt{(1 - x^2)^2 + 4(1 - \gamma^{-2})}}{1 - \gamma^{-2}} > 0, \quad \gamma \neq 1.$$

If $\gamma = 1$, then $K_2(x) = -\frac{C^2(x)}{2A(x)} = -\frac{4}{2(1-x^2)}$. To fulfill condition $K_2 > 0$, condition $A(x) = 1 - x^2 < 0$ that defines set

$|x| < 1$ of possible functioning of the system must be fulfilled.

Modeling. For modeling under different initial conditions, finite time interval $T = [0, 20]$ was selected, since all transient processes in a closed system are practically completed.

According to Figures 1–3, the value of the state vector asymptotically tends to zero for different initial conditions, which indicates the stability of the system and the correct selection of parameters, under which the system retains the property of stability subject to any given limited disturbances.

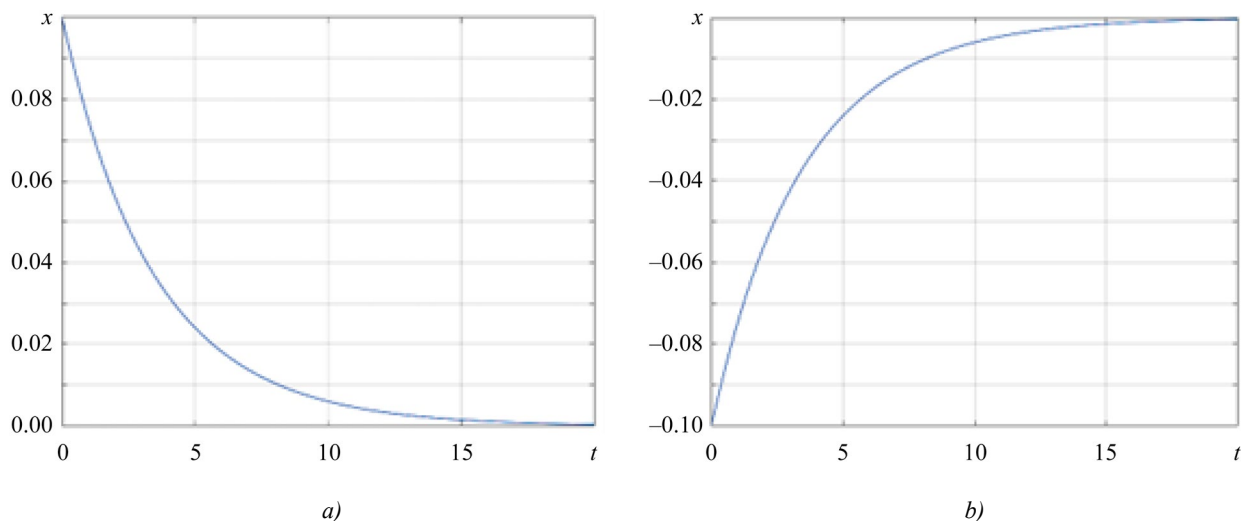


Fig. 1. Change of the state vector:
a — for initial state $x_0 = 0.1$; b — for initial state $x_0 = -0.1$

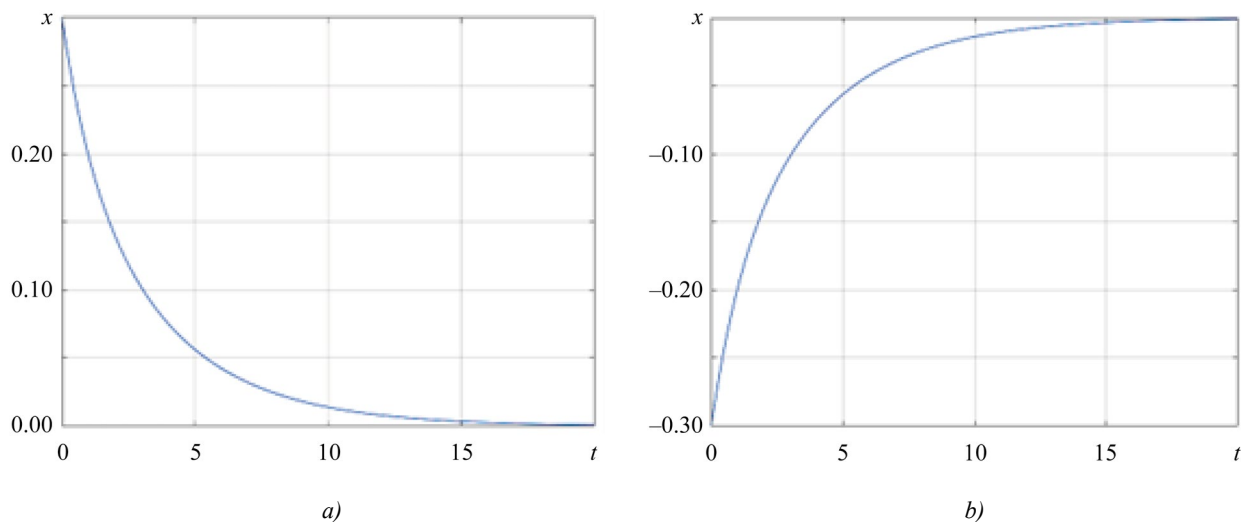


Fig. 2. Change of the state vector:
a — for initial state $x_0 = 0.3$; b — for initial state $x_0 = -0.3$

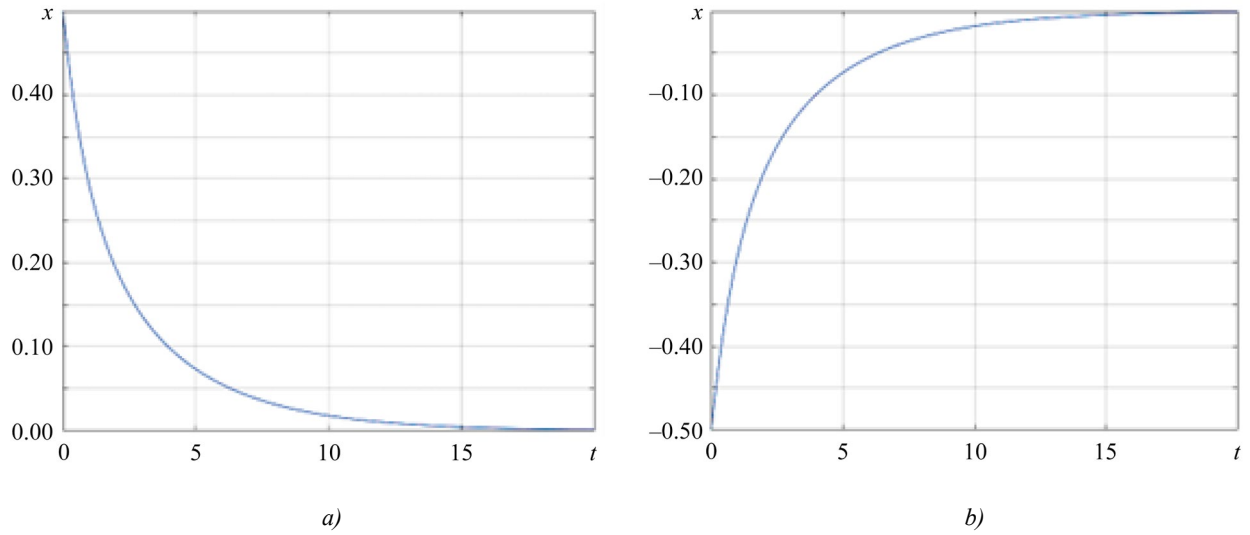


Fig. 3. Change of the state vector:
 a — for initial state $x_0 = 0.5$; b — for initial state $x_0 = -0.5$

Model example No. 2. One of the options of the two-dimensional case is considered, when equations (1), (2) and functional (5) have the form:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ x_1 & 0 \end{pmatrix}}_{A(x)} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_1(x)} w(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_2(x)} u(t),$$

$$y(t) = x(t),$$

$$I(u, w) = \int_0^\infty \left[x^T(t) x(t) + u^2(t) - \gamma^2 w^2(t) \right] dt \leq 0,$$

i.e., $C(x) = E_2$, $S(x) = E_2$, $Q(x) = 1$, $P(x) = 1$.

Solution. The control structures of the object and the disturbance follow from (18):

$$u^*(x) = - \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -K_{12}(x)x_1 - K_{22}(x)x_2,$$

$$w^*(x) = \frac{1}{\gamma^2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{\gamma^2} [K_{12}(x)x_1 + K_{22}(x)x_2],$$

and equation (17) has the form:

$$\begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} -$$

$$- \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} - \frac{1}{\gamma^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \right] \begin{pmatrix} K_{11}(x) & K_{12}(x) \\ K_{12}(x) & K_{22}(x) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From here,

$$2K_{12}x_1 - \delta K_{12}^2 + 1 = 0,$$

$$K_{11} + K_{22}x_1 - \delta K_{12}K_{22} = 0,$$

$$2K_{12} - \delta K_{22}^2 + 1 = 0,$$

where $\delta = \frac{\gamma^2 - 1}{\gamma^2} = 1 - \gamma^{-2}$.

Solution to the first equation $\delta K_{12}^2 - K_{12}x_1 - 1 = 0$ has the form:

$$K_{12} = \frac{2x_1 \pm \sqrt{4x_1^2 + 4\delta}}{2\delta} = \frac{x_1 \pm \sqrt{x_1^2 + \delta}}{\delta}.$$

Solution to the third equation:

$$K_{22} = \sqrt{\frac{2K_{12} + 1}{\delta}} = \sqrt{\frac{2x_1 + \delta \pm 2\sqrt{x_1^2 + \delta}}{\delta^2}}.$$

Solution to the second equation:

$$K_{11} = K_{22}(\delta K_{12} - x_1) = \sqrt{\frac{2x_1 + \delta \pm 2\sqrt{x_1^2 + \delta}}{\delta^2}} \pm \sqrt{x_1^2 + \delta}.$$

In the obtained solutions, positive signs were selected taking into account condition $K_2 > 0$.

Modeling. To model the system, it is required to select the minimum possible value γ , so that it satisfies (4) and at the same time guarantees the asymptotic stability of the closed system. In example 1, the value of parameter γ was found analytically, but a problem arose with determining the value of the optimal parameter γ^* , therefore value γ , used in the modeling, was selected experimentally. When $\gamma^* = 1.5$, the system remained stable according to expression (19). For modeling, the time interval $T = [0, 10]$ was selected, since transient processes in the closed system decayed fairly quickly. Modeling was performed for various initial conditions.

According to Figures 4–5, it can be concluded that the coordinates of the state vector asymptotically tend to zero. This result is observed for each of the initial conditions considered. This indicates that the system is stable, and its parameters are selected correctly, which allows maintaining the property of stability under any initial conditions and under the worst disturbances.

The initial conditions significantly affect the trajectories of the change in the coordinates of the state vector, but from Figures 4–5, it is clear that the proposed approach allows us not only to compensate for external disturbances, but also to stabilize the trajectory of motion.

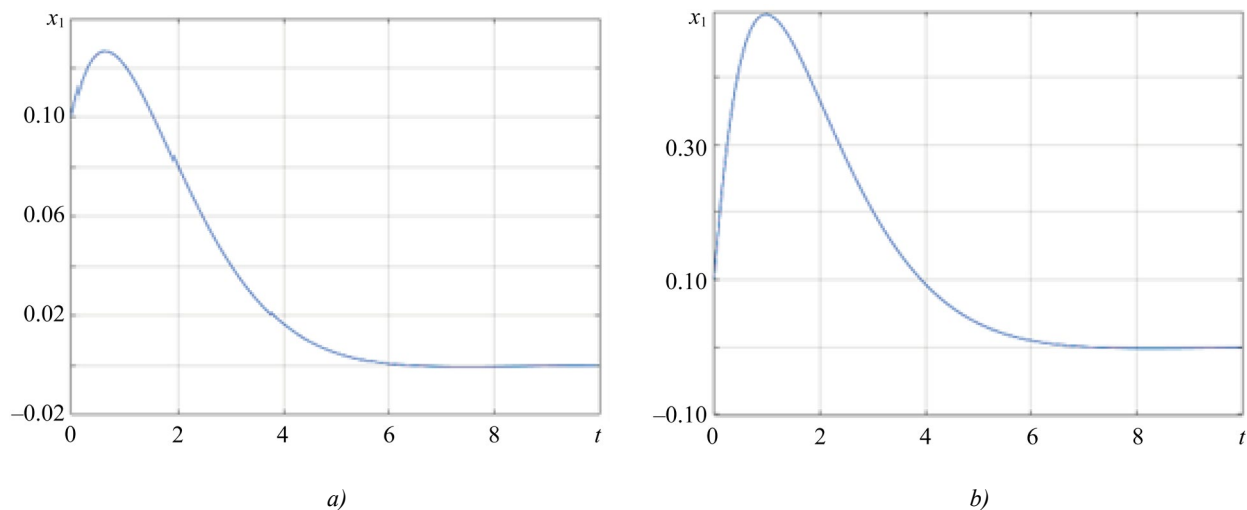


Fig. 4. Change of $x_1(t)$:
a — for initial state $x_0 = (0, 1; 0, 1)^T$; b — for initial state $x_0 = (0, 1; 1)^T$

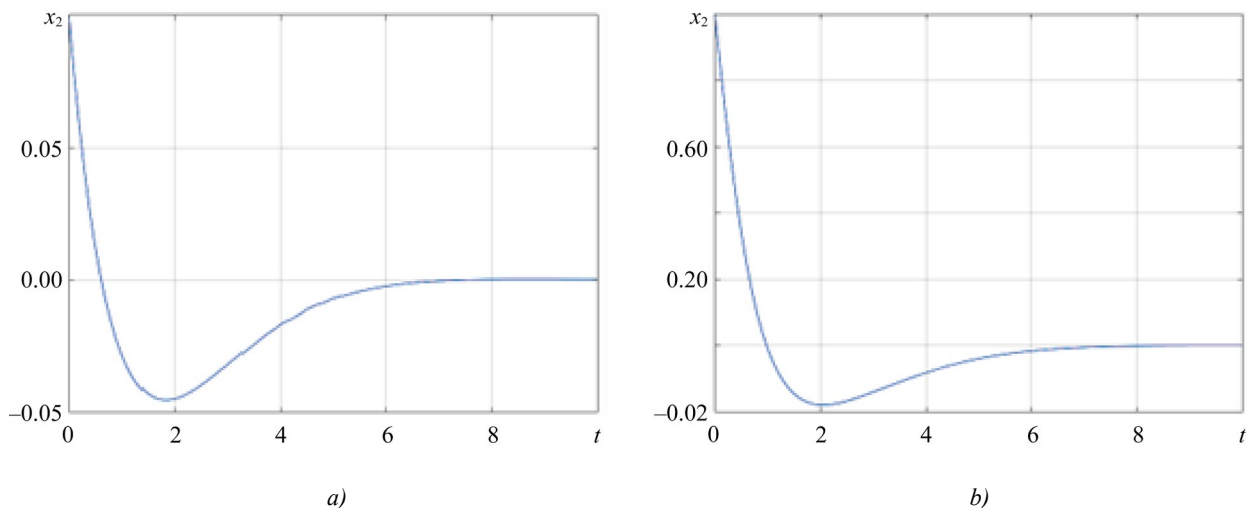


Fig. 5. Change of $x_2(t)$:
a — for initial state $x_0 = (1; 0, 1)^T$; b — for initial state $x_0 = (1; 1)^T$

Discussion of the Results. As a result of the study, sufficient H_∞ – control conditions were formulated and substantiated, an approximate solution method was developed. The method proposed within the framework of the problem was tested on two model examples. The simulation results allow us to conclude that the use of the developed controller synthesis method guarantees the required quality of transient processes and provides asymptotic stability of closed systems.

Conclusion. The results and methods proposed in this paper can be applied to solve control problems of varying complexity — from designing simple autopilots to developing complex automatic navigation systems for manned and unmanned aerial vehicles. This emphasizes the prospects of using the proposed approach and makes it an attractive option for further research.

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Conflict of Interest Statement: the authors declare no conflict of interest.

All authors have read and approved the final manuscript.

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Заявленный вклад авторов:

А.В. Пантелеев: разработка концепции, научное руководство, разработка методологии, написание рукописи.

А.А. Яковлева: разработка методологии и программного обеспечения, написание черновика рукописи, визуализация.

Конфликт интересов: авторы заявляют об отсутствии конфликта интересов.

Все авторы прочитали и одобрили окончательный вариант рукописи.

Received / Поступила в редакцию 30.04.2025

Reviewed / Поступила после рецензирования 24.05.2025

Accepted / Принята к публикации 28.05.2025