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1/3 SUBHARMONIC RESPONSE OF DUFFING OSCILLATOR UNDER PERIODIC AND RANDOM EXCITATIONS*

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The subharmonic response of one third order of Duffing oscillator under harmonic and random excitations is investigated for the first time by a technique combining the stochastic averaging method, the equivalent linearization method, and the technique of auxiliary function for Fokker-Planck equation. The averaged equations are linear zed so that the stationary density function of the approximate response can be found exactly by the technique of auxiliary function. The one third order subharmonic response obtained by the present technique is validated by numerical simulation. The significant contribution of this work is that it may lead to a new trend in investigating subharmonic oscillators in random nonlinear systems.

Keywords: *Duffing oscillator, subharmonic, averaging method, equivalent linearization, auxiliary function, harmonic excitation, random excitation.*

Introduction. In this paper, we are concerned with the Duffing oscillator, which has been applied to model many mechanical systems and has attracted much attention as a typical nonlinear system. When the system is under only a harmonic excitation or random one, two popular tools used to study such a nonlinear system are the averaging method and equivalent linearization method, respectively. The former was originally given by Krylov and Bogolyubov [1] and then it was developed by Bogolyubov and Mitropolskiy [2-4] and was extended to systems under a random excitation with the works of Stratonovich [5], Khasminskii [6], and others, which were reviewed in survey paper by Mitropolskiy [3], Robert and Spanos [7] and Manohar [8]. The later, the stochastic equivalent linearization method, which has attracted many researchers due to its originality and capability for various applications in engineering, was first studies by Kazakov [9], who extended Krylov and Bogolyubov's linearization technique [1] of deterministic problems to random problems. This method was also reviewed in some books by Roberts and Spanos [10], and Socha [11]. Recently, some approaches to the stochastic linearization have been proposed in Refs. [12-14]. In [13-14], for example, Anh *et al.* have proposed a dual criterion of stochastic linearization method for single and multi-degree-of-freedom nonlinear systems under white noise random excitations. The authors showed that the accuracy of the mean-square response is significantly improved when the nonlinearity increases.

In a Duffing oscillator under periodic excitation, the phenomenon of subharmonic response has been known for years and has been described in many textbooks (see e.g. [15-18]) and works (see e.g. [19-21]). When the system is subjected to a combination of harmonic and random excitations, however, to the authors' knowledge, although the response of this oscillator has received a flurry of research effort for years (see e.g. [22-25]), there is no work on its subharmonic response. Thus, in this research, we present a technique to treat a one third order subharmonic response of a Duffing oscillator subjected to periodic and random excitations. The technique is a combination of the stochastic averaging method, the equivalent linearization method, and the technique of auxiliary function which yields the exact joint stationary probability den-

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sity function (PDF) for the equivalent linear system [4,26,27]. The approximate analytical solution of the Duffing system obtained by the proposed technique is validated by numerical simulation results, obtained by Monte-Carlo method.

1. Formulation problem. Let us consider Duffing oscillator under harmonic and random excitations of the form

$$\ddot{z}+\varepsilon h\dot{z}+\varepsilon\gamma z^3+\omega^2 z=P\cos\nu t+\sqrt{\varepsilon}\,\sigma\dot{\xi}(t), \tag{1}$$

where $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$ are the displacement, velocity and acceleration of the system, respectively; ε is a small positive parameter; h is the damping coefficient; γ is the nonlinear stiffness coefficient; ω is the natural frequency of the corresponding linear system when $\varepsilon=0$; P, ν and σ are parameters; and function $\dot{\xi}(t)$ is a Gaussian white noise process of unit intensity with the correlation function $R_{\dot{\xi}}(\tau)=E\left[\dot{\xi}(t)\dot{\xi}(t+\tau)\right]=\delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function, and notation $E(.)$ denotes the mathematical expectation operator. It is supposed that the natural frequency ω is close to $\nu/3$, i.e. parameters ω and ν have the relation

$$\omega^2-\frac{\nu^2}{9}=\varepsilon\Delta, \tag{2}$$

where Δ is a detuning parameter. We introduce a new variable x as follows

$$x=z-Q\cos\nu t,\ Q=\frac{P}{\omega^2-\nu^2}. \tag{3}$$

Using (3) Eq. (1) can be rewritten in the form

$$\ddot{x}+\omega^2 x=\varepsilon\left[-h\left(\dot{x}-\nu Q\sin\nu t\right)-\gamma\left(x+Q\cos\nu t\right)^3\right]+\sqrt{\varepsilon}\sigma\dot{\xi}(t). \tag{4}$$

Substituting (2) into Eq. (4) we obtain

$$\ddot{x}+\left(\frac{\nu}{3}\right)^2 x=\varepsilon F(x,\dot{x},t)+\sqrt{\varepsilon}\sigma\dot{\xi}(t), \tag{5}$$

where

$$F(x,\dot{x},t)=-\Delta x-h\left(\dot{x}-\nu Q\sin\nu t\right)-\gamma\left(x+Q\cos\nu t\right)^3. \tag{6}$$

We seek the solution of Eq. (5) in the form of

$$x=b\cos\frac{\nu}{3}t+d\sin\frac{\nu}{3}t,\ \dot{x}=-\frac{b\nu}{3}\sin\frac{\nu}{3}t+\frac{d\nu}{3}\cos\frac{\nu}{3}t, \tag{7}$$

where b and d are slowly varying random processes satisfying an additional condition

$$\dot{b}\cos\frac{\nu}{3}t+\dot{d}\sin\frac{\nu}{3}t=0. \tag{8}$$

Substituting (7) into Eq. (5) and then solving the resulting equation and Eq. (8) with respect to the derivatives $\dot{b}$ and $\dot{d}$ yield

$$\begin{aligned}\dot{b}&=-\frac{3}{\nu}\left(\varepsilon F+\sqrt{\varepsilon}\sigma\dot{\xi}(t)\right)\sin\frac{\nu}{3}t,\\ \dot{d}&=\frac{3}{\nu}\left(\varepsilon F+\sqrt{\varepsilon}\sigma\dot{\xi}(t)\right)\cos\frac{\nu}{3}t,\end{aligned} \tag{9}$$

where, noting (6) and (7),

$$F = -\Delta\left(b\cos\frac{\nu}{3}t + d\sin\frac{\nu}{3}t\right) - h\left(-\frac{b\nu}{3}\sin\frac{\nu}{3}t + \frac{d\nu}{3}\cos\frac{\nu}{3}t - \nu Q\sin\nu t\right)$$
$$-\gamma\left(b\cos\frac{\nu}{3}t + d\sin\frac{\nu}{3}t + Q\cos\nu t\right)^3. \quad (10)$$

The pair of stochastic differential equations (9) can be simplified by using the stochastic averaging method [3-7]

$$\dot{b} = \varepsilon H_1(b,d) + \frac{3\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\dot{\xi}_1(t),$$
$$\dot{d} = \varepsilon H_2(b,d) + \frac{3\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\dot{\xi}_2(t). \quad (11)$$

Here $\dot{\xi}_1(t)$ and $\dot{\xi}_2(t)$ are independent white noises with unit intensity, and the drift coefficients $H_1(b,d)$ and $H_2(b,d)$ are determined as follows

$$H_1(b,d) = -\frac{1}{2\pi}\int_0^{2\pi}\frac{3}{\nu}F\sin\frac{\nu}{3}t d\left(\frac{\nu}{3}t\right),\ H_2(b,d) = \frac{1}{2\pi}\int_0^{2\pi}\frac{3}{\nu}F\cos\frac{\nu}{3}t d\left(\frac{\nu}{3}t\right). \quad (12)$$

Substituting(6), (7) into (12) yields the drift coefficients of the system (11)

$$H_1(b,d) = -\frac{h}{2}b + \frac{6\Delta + 9\gamma Q^2}{4\nu}d + \frac{9\gamma}{8\nu}\left(b^2d + d^3 - 2Qbd\right),$$
$$H_2(b,d) = -\frac{6\Delta + 9\gamma Q^2}{4\nu}b - \frac{h}{2}d + \frac{9\gamma}{8\nu}\left(-b^3 - Qb^2 - bd^2 + Qd^2\right). \quad (13)$$

The Fokker-Planck (FP) equation written for the stationary probability density function (PDF) $W(b,d)$ associated with the system (11) has the form

$$\frac{\partial}{\partial b}\left(H_1(b,d)W\right) + \frac{\partial}{\partial d}\left(H_2(b,d)W\right) = \frac{9\sigma^2}{4\nu^2}\left[\frac{\partial^2}{\partial b^2}(W) + \frac{\partial^2}{\partial d^2}(W)\right]. \quad (14)$$

Solution of (14) is still a difficult problem so far because functions $H_1(b,d)$ and $H_2(b,d)$ are nonlinear functions in b,d. To overcome this, the equivalent linearization method is employed. Following this method, the nonlinear functions H_1, H_2 are replaced by linear ones. Noting (13), we denote

$$g_1(b,d) = \frac{9\gamma}{8\nu}\left(b^2d + d^3 - 2Qbd\right),$$
$$g_2(b,d) = \frac{9\gamma}{8\nu}\left(-b^3 - Qb^2 - bd^2 + Qd^2\right). \quad (15)$$

According to the stochastic equivalent linearization method the nonlinear functions (15) are replaced by

$$\bar{g}_1(b,d) = \eta_{11}b + \eta_{12}d + \eta_{13},$$
$$\bar{g}_2(b,d) = \eta_{21}b + \eta_{22}d + \eta_{23}, \quad (16)$$

where linearization coefficients $\eta_{ij}, i = 1,2; j = 1,2,3$ are to be determined by an optimization criterion. Thus, the functions $H_i, i = 1,2$ in (13) are replaced by linear functions

$$H_1(b,d)=\left(-\frac{h}{2}+\eta_{11}\right)b+\left(\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{12}\right)d+\eta_{13},$$
$$H_2(b,d)=\left(-\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{21}\right)b+\left(-\frac{h}{2}+\eta_{22}\right)d+\eta_{23}. \quad (17)$$

According to the technique of auxiliary function with the constant auxiliary function taking the form (see [4,26,27] for details)

$$u_0=\frac{9\sigma^2}{4\nu^2}\frac{-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{12}}{-h+\eta_{11}+\eta_{22}}, \quad (18)$$

the corresponding FP equation to Eq. (14), where drift coefficients are linear functions (17), has the following exact solution

$$W(b,d)=C\exp\left\{-\tau_1 b^2-\tau_2 d^2+\tau_3 bd+\tau_4 b+\tau_5 d\right\}, \quad (19)$$

where C is a normalization constant and coefficients $\tau_i, i=\overline{1,5}$ are determined as follows

$$\tau_1=-\Psi\left(\left(-\frac{h}{2}+\eta_{11}\right)\left(-h+\eta_{11}+\eta_{22}\right)+\left(-\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{21}\right)\left(-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{12}\right)\right),$$

$$\tau_2=-\Psi\left(\left(-h+\eta_{11}+\eta_{22}\right)\left(-\frac{h}{2}+\eta_{22}\right)-\left(-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{12}\right)\left(\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{12}\right)\right),$$

$$\tau_3=2\Psi\left(\left(-\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{21}\right)\left(-\frac{h}{2}+\eta_{22}\right)+\left(\frac{6\Delta+9\gamma Q^2}{4\nu}+\eta_{12}\right)\left(-\frac{h}{2}+\eta_{11}\right)\right), \quad (20)$$

$$\tau_4=2\Psi\left(\eta_{13}\left(-h+\eta_{11}+\eta_{22}\right)+\eta_{23}\left(-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{12}\right)\right),$$

$$\tau_5=2\Psi\left(\left(-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{21}\right)\eta_{13}+\left(-h+\eta_{11}+\eta_{22}\right)\eta_{23}\right),$$

where

$$\Psi=\frac{2\nu^2\left(-h+\eta_{11}+\eta_{22}\right)}{9\sigma^2\left[\left(-\frac{6\Delta+9\gamma Q^2}{2\nu}+\eta_{21}-\eta_{12}\right)^2+\left(-h+\eta_{11}+\eta_{22}\right)^2\right]}. \quad (21)$$

It is noted that the joint PDF $W(b,d)$ determined by (19) has finite integral if coefficients τ_1 and τ_2 are positive. Therefore, the approximate stationary PDF of Eq. (14) is determined by (19) whose coefficients are given in (20). It is seen from (19) that random variables b and d are jointly Gaussian. Thus, from(19), one obtains

$$E(b)=\frac{2\tau_2\tau_4+\tau_3\tau_5}{4\tau_1\tau_2-\tau_3^2},\ E(d)=\frac{2\tau_1\tau_5+\tau_3\tau_4}{4\tau_1\tau_2-\tau_3^2},\ \sigma_b^2=\frac{2\tau_2}{4\tau_1\tau_2-\tau_3^2}, \quad (22)$$

$$\sigma_d^2=\frac{2\tau_1}{4\tau_1\tau_2-\tau_3^2},\ k_{bd}=\frac{\tau_3}{4\tau_1\tau_2-\tau_3^2},$$

where σ_b^2 and σ_d^2 are variance of b and d, respectively, and k_{bd} is covariance of b and d. It is seen from (22) that necessary statistics of processes b and d can be computed algebraically based on coefficients of joint PDF $W(b,d)$. Thus, the approximate solution (19) of Eq. (1) is completely determined when the linearization coefficients $\eta_{ij}, i=1,2; j=1,2,3$ are found.

There are some criteria for determining the coefficients η_{ij} [20]. In this work, we use the mean square error criterion which requires that the mean square of the following errors be minimum [10,11]. From (13), (15)-(17) we have the errors when using linearization method to be

$$e_i = g_i(b,d) - (\eta_{i1}b + \eta_{i2}d + \eta_{i3}), i=1,2. \tag{23}$$

So, the mean square error criterion leads to

$$E(e_i^2) = E\left\{\left[g_i(b,d) - (\eta_{i1}b + \eta_{i2}d + \eta_{i3})\right]^2\right\} \to \min_{\eta_{ij}}, \ i=1,2; \ j=1,2,3. \tag{24}$$

From

$$\frac{\partial}{\partial \eta_{ij}} E(e_i^2) = 0, \ i=1,2; j=1,2,3, \tag{25}$$

it follows that

$$\begin{aligned}
&E\left[b\, g_1(b,d)\right] - E(b^2)\eta_{11} - E(bd)\eta_{12} - E(b)\eta_{13} = 0,\\
&E\left[d\, g_1(b,d)\right] - E(bd)\eta_{11} - E(d^2)\eta_{12} - E(d)\eta_{13} = 0,\\
&E\left[g_1(b,d)\right] - E(b)\eta_{11} - E(d)\eta_{12} - \eta_{13} = 0,\\
&E\left[b\, g_2(b,d)\right] - E(b^2)\eta_{21} - E(bd)\eta_{22} - E(b)\eta_{23} = 0,\\
&E\left[d\, g_2(b,d)\right] - E(bd)\eta_{21} - E(d^2)\eta_{22} - E(d)\eta_{23} = 0,\\
&E\left[g_2(b,d)\right] - E(b)\eta_{21} - E(d)\eta_{22} - \eta_{23} = 0,
\end{aligned} \tag{26}$$

where $g_1(b,d)$, $g_2(b,d)$ are given by (15). Using the fact that b and d are jointly Gaussian, all higher moments of b and d in (26) can be expressed in terms of the first and second moments of b and d by the following properties of a Gaussian random vector $\vec{X} = (X_1, X_2) = (b,d)$ [28]

$$\begin{aligned}
&E(X_i^{n+1}) = E(X_i)E(X_i^n) + n\sigma_{X_i}^2 E(X_i^{n-1}),\\
&E(X_i X_1^{n_1} X_2^{n_2}) = E(X_i)E(X_1^{n_1} X_2^{n_2}) + n_1 k_{X_i X_1} E(X_1^{n_1-1} X_2^{n_2}) + n_2 k_{X_i X_2} E(X_1^{n_1} X_2^{n_2-1}), \ i=1,2.
\end{aligned} \tag{27}$$

Solving system (26) in $\eta_{ij}, i=1,2; j=1,2,3$ with noting (27) gives

$$\begin{aligned}
\eta_{11} &= \frac{9\gamma}{4\nu}\left(k_{bd} + E(b)E(d) - QE(d)\right),\\
\eta_{12} &= \frac{9\gamma}{8\nu}\left(\sigma_b^2 + E^2(b) + 3\sigma_d^2 + 3E^2(d) - 2QE(b)\right),\\
\eta_{13} &= -\frac{9\gamma}{4\nu}\left(Q\,k_{bd} + E^2(b)E(d) - QE(b)E(d) + E^3(d)\right),\\
\eta_{21} &= -\frac{9\gamma}{8\nu}\left(3\sigma_b^2 + 3E^2(b) + \sigma_d^2 + E^2(d) + 2QE(b)\right),\\
\eta_{22} &= -\frac{9\gamma}{4\nu}\left(k_{bd} + E(b)E(d) - QE(d)\right),\\
\eta_{23} &= -\frac{9\gamma}{8\nu}\left(Q\sigma_b^2 - 2E^3(b) - QE^2(b) + QE^2(d) - 2E(b)E^2(d) - Q\sigma_d^2\right).
\end{aligned} \tag{28}$$

Thus, $\eta_{ij}, i=1,2; j=1,2,3$ are determined from the closed system of eleven equations for eleven unknowns η_{ij}, $i=1,2; j=1,2,3$; $E(b), E(d), \sigma_b^2, \sigma_d^2, k_{bd}$, obtained by combining equations in (20), (22), and (28). After being found by solving this closed system, the values of coefficients η_{ij}, $i=1,2$; $j=1,2,3$ are to be substituted into (19) to obtain the approximate stationary PDF in b and d of Duffing equation (1).

From (7), the mean square response of Eq. (5) can be determined as follows

$$E\left[x^2(t)\right]=E\left(b^2\right)\cos^2\frac{\nu}{3}t+E\left(d^2\right)\sin^2\frac{\nu}{3}t+E(bd)\sin\frac{2\nu}{3}t. \tag{29}$$

Taking averaging with respect to time Eq. (29) gives

$$\left\langle E\left[x^2(t)\right]\right\rangle_t=\frac{1}{2\pi}\int_0^{2\pi}E\left[x^2(t)\right]d\left(\frac{\nu}{3}t\right)=\frac{1}{2}\left(E\left(b^2\right)+E\left(d^2\right)\right). \tag{30}$$

From properties of a variance of a random variable [28], Eq. (30) can be rewritten as

$$\left\langle E\left[x^2(t)\right]\right\rangle_t=\frac{1}{2}\left(E^2(b)+\sigma_b^2+E^2(d)+\sigma_d^2\right). \tag{31}$$

Substituting (22) into (31) and reducing the obtained result yield the time-averaging of mean square response to be

$$\left\langle E\left[x^2(t)\right]\right\rangle_t=\frac{\left(2\tau_2\tau_4+\tau_3\tau_5\right)^2+\left(2\tau_1\tau_5+\tau_3\tau_4\right)^2}{2\left(4\tau_1\tau_2-\tau_3^2\right)^2}+\frac{\tau_1+\tau_2}{4\tau_1\tau_2-\tau_3^2}, \tag{32}$$

where $\tau_i, i=\overline{1,5}$ are given by (20). From (3), one obtains

$$\left\langle E\left(z^2\right)\right\rangle_t=\frac{1}{2\pi}\int_0^{2\pi}E\left[\left(x(t)+Q\cos\nu t\right)^2\right]d\left(\frac{\nu t}{3}\right)=\left\langle E\left[x^2(t)\right]\right\rangle_t+\frac{Q^2}{2}. \tag{33}$$

Substituting (32) into (33) yields

$$\left\langle E\left(z^2\right)\right\rangle_t=\frac{\left(2\tau_2\tau_4+\tau_3\tau_5\right)^2+\left(2\tau_1\tau_5+\tau_3\tau_4\right)^2}{2\left(4\tau_1\tau_2-\tau_3^2\right)^2}+\frac{\tau_1+\tau_2}{4\tau_1\tau_2-\tau_3^2}+\frac{Q^2}{2}. \tag{34}$$

This formula shows that the time-averaged mean square one third order subharmonic response of the system can be computed from the coefficients of the stationary PDF (19).

2. Numerical results. In the numerical simulation, the parameters in system (1) are chosen as follows $\omega=1$, $\nu=3.01$, $\varepsilon=0.01$, $\gamma=1$, $h=2$, $P=1$. The various values of the subharmonic response of Duffing equation (1) are compared to the numerical simulation results versus the parameter σ^2. The numerical simulation of the time- averaged mean square response $\left\langle z^2\right\rangle_{sim}$ is obtained by 10,000-realization Monte Carlo simulation with time t in the interval (900s, 1000s). The time-averaged mean square response $\left\langle E\left(z^2\right)\right\rangle_t$ of the Duffing oscillator, obtained by the system (22), (28) and (34), are compared to a numerical result. The response versus the parameter σ^2 is evaluated in Table 1 where the error is defined as

$$\text{Err}=\frac{\left|\left\langle z^2\right\rangle_{sim}-\left\langle E\left(z^2\right)\right\rangle_t\right|}{\left\langle z^2\right\rangle_{sim}}\times 100\%. \tag{35}$$

It is seen from Table 1 that the proposed technique gives a good prediction. Moreover, Fig. 1 portrays the variation of the mean square subharmonic responses $\left\langle E\left(z^2\right)\right\rangle_t$ obtained by the present technique with the noise parameter σ^2 compared to ones obtained by Monte-Carlo simulation. It is seen that the theoretical prediction and the simulations agree very well.

Table 1

The error between the simulation result and approximate values of the time-averaging of mean square response $\langle z^2(t)\rangle_t$ versus the parameter σ^2 $(\omega = 1, \nu = 3.01, P = 1, h = 2, \varepsilon = 0.01, \gamma = 1)$

σ^2	$\langle z^2\rangle_{sim}$	$\langle E(z^2)\rangle_t$	Err(%)
0.1	0.0323	0.0331	2.48
0.5	0.1328	0.1350	1.69
1	0.2599	0.2626	1.03
2	0.4978	0.5136	3.18
3	0.7272	0.7607	4.61
4	0.9850	1.0074	2.28
5	1.2271	1.2545	2.23

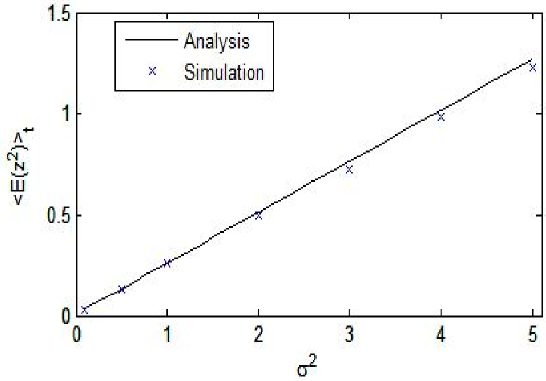

Fig. 1. The time-averaged mean square subharmonic response versus the parameter σ^2 $(\omega = 1, \nu = 3.01,\ P = 1, h = 2,\ \varepsilon = 0.01, \gamma = 1)$

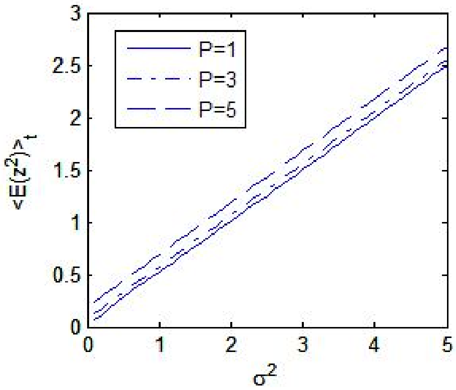

Fig. 2. Effects of σ^2 and P on the mean square subharmonic response, $\omega = 1, \nu = 3.01, \varepsilon = 0.01, h = 1, \gamma = 1$

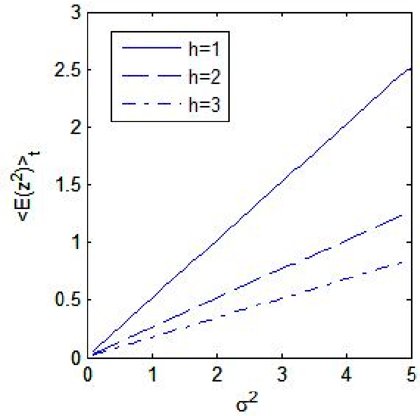

Fig. 3. Effects of σ^2 and h on the mean square subharmonic response, $\omega = 1, \nu = 3.01, \varepsilon = 0.01, P = 1, \gamma = 1$

Next, we investigate the effects of the noise intensity σ^2, external force amplitude P, and the damping term h on the one third order subharmonic response based on equations (22), (28) and (34). With initial values

$$(\eta_{11}, \eta_{12}, \eta_{13}, \eta_{21}, \eta_{22}, \eta_{23}) = (-1, -1, 1, 0, -1, 1),$$

and the input parameters $\omega = 1$, $\nu = 3.01$, $\varepsilon = 0.01$, $\gamma = 1$, theoretical results are shown in Fig. 2 and Fig. 3. It can be observed that from Fig. 2 that the mean response amplitude increases when harmonic excitation increases. In Fig. 3, we can see that for given parameters ω, ν, P, ε, γ the time-averaged mean square response decreases as the damping coefficient increases.

3. Summary and conclusions. The averaging method and the equivalent linearization method are famous tools in studying nonlinear systems subjected to harmonic and random excitation, respectively. A combination of those methods will give a power tool to study complex systems. In this work, the subharmonic response of one third order of Duffing oscillator under a combination of harmonic and random excitations is investigated. The technique used in our research is a combination of the two famous methods mentioned above and the technique of auxiliary function to overcome the difficulty in solving the corresponding FP equation. The key steps of the technique are summarized as follows. First, the stochastic averaging of the equation (1) is carried out in Cartesian coordinates by the transformations (3) and (7). The drift coefficients of the averaged equations in the system (11) are polynomial forms in random variables which give an advantageous context to apply the equivalent linearization method. The linearization coefficients are determined by a closed system including the equations in (22), (28) and (34). The FP equation associated with the equivalent linearized system can be solved exactly by the technique of auxiliary function. The theoretical results are agreed well with numerical simulations.

It appears that the new approach gained through this study has a large potential and it will become helpful for other types of nonlinear systems.

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Субгармонический отклик третьего порядка для осциллятора Дуффинга, возмущенного гармоническим и случайным воздействием*

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В статье впервые исследуется субгармонический отклик третьего порядка осциллятора Дуффинга на основе метода стохастического усреднения и одновременно стохастической линеаризации. При этом используется разрабатываемый авторами метод вспомогательных функций для уравнения Фоккера – Планка. Усредненные уравнения линеаризованы так, что плотностная стационарная функция приближенного отклика может быть получена точно с помощью метода вспомогательной функции. Полученные на основе разработанного метода решения сравниваются с численными решениями. Значение этой работы заключается в том, что предложенный метод может привести к новой тенденции в исследовании субгармонических осцилляторов в случайных нелинейных систем.

Ключевые слова: *осциллятор Дюффинга, субгармоника, метод усреднения, эквивалентная линеаризация, вспомогательная функция, гармонические возбуждения, случайные возбуждения.*

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